JumpDiff

Release 0.4.1

Aug 02, 2021

Contents

`JumpDiff` is a `python` library with non-parametric Nadaraya—Watson estimators to extract the parameters of jump-diffusion processes. With *JumpDiff* one can extract the parameters of a jump-diffusion process from one-dimensional timeseries, employing both a kernel-density estimation method combined with a set on second-order corrections for a precise retrieval of the parameters for short timeseries.

CHAPTER 1

Installation

To install `jumpdiff` simply use

```
pip install jumpdiff
```

Then on your favourite editor just use

```
import jumpdiff as jd
```

The library depends on `numpy`, `scipy`, and `sympy`.

CHAPTER 2

Jump-diffusion processes

2.1 The theory

Jump-diffusion processes[1], as the name suggest, are a mixed type of stochastic processes with a diffusive and a jump term. One form of these processes which is mathematically traceable is given by the Stochastic Differential Equation

$$\mathrm{d}X(t) = a(x,t)\ \mathrm{d}t + b(x,t)\ \mathrm{d}W(t) + \xi\ \mathrm{d}J(t),$$

which has four main elements: a drift term $a(x,t)$, a diffusion term $b(x,t)$, linked with a Wiener process $W(t)$, a jump amplitude term $\xi(x,t)$, which is given by a Gaussian distribution $\mathcal{N}(0,\sigma_\xi^2)$ coupled with a jump rate λ, which is the rate of the Poissonian jumps $J(t)$. You can find a good review on this topic in Ref. 2.

2.2 Integrating a jump-diffusion process

Let us use the functions in `jumpdiff` to generate a jump-difussion process, and subsequently retrieve the parameters. This is a good way to understand the usage of the integrator and the non-parametric retrieval of the parameters.

First we need to load our library. We will call it `jd`

```
import jumpdiff as jd
```

Let us thus define a jump-diffusion process and use `jd_process` to integrate it. Do notice here that we need the drift $a(x,t)$ and diffusion $b(x,t)$ as functions.

```
# integration time and time sampling
t_final = 10000
delta_t = 0.001

# A drift function
def a(x):
    return -0.5*x
```

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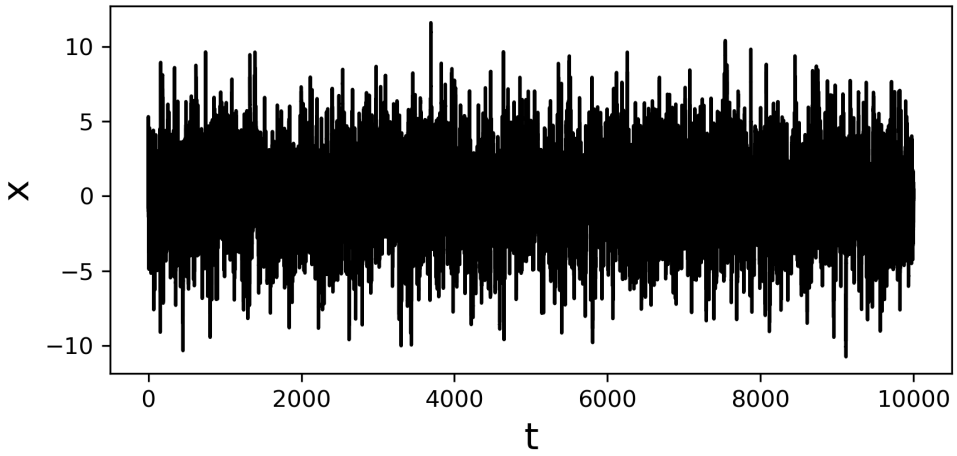
```
# and a (constant) diffusion term
def b(x):
    return 0.75

# Now define a jump amplitude and rate
xi = 2.5
lamb = 1.75

# and simply call the integration function
X = jd.jd_process(t_final, delta_t, a=a, b=b, xi=xi, lamb=lamb)
```

This will generate a jump diffusion process `X` of length `int(10000/0.001)` with the given parameters.



2.3 Using `jumpdiff` to retrieve the parameters

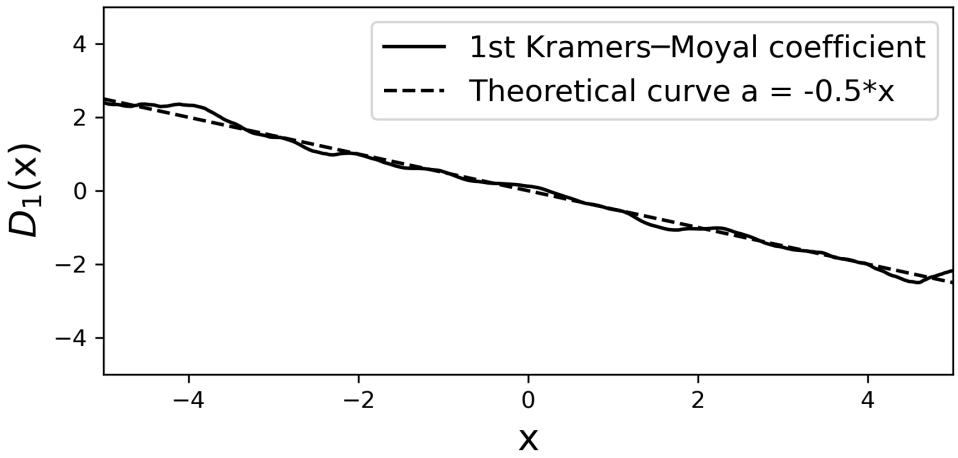
2.3.1 Moments and Kramers–Moyal coefficients

Take the timeseries `X` and use the function `moments` to retrieve the conditional moments of the process. For now let us focus on the shortest time lag, so we can best approximate the Kramers—Moyal coefficients. For this case we can simply employ

```
edges, moments = jd.moments(timeseries = X)
```

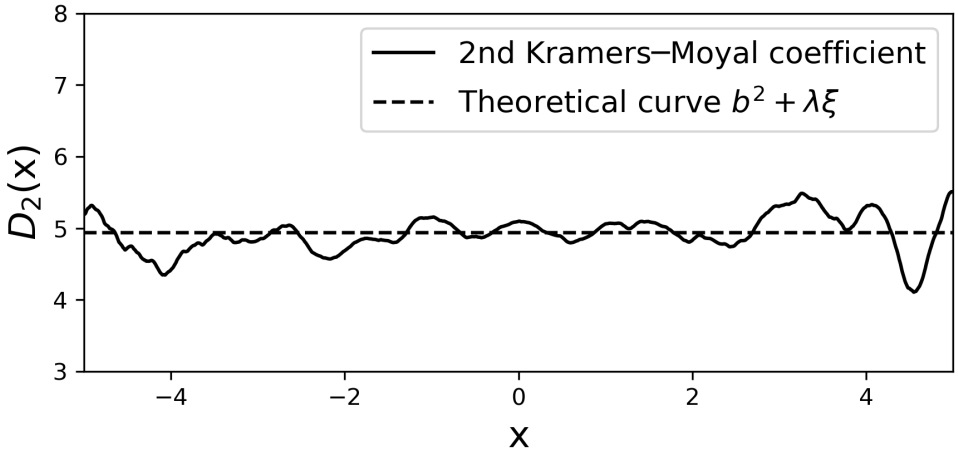
In the array `edges` are the limits of our space, and in our array `moments` are recorded all 6 powers/order of our conditional moments. Let us take a look at these before we proceed, to get acquainted with them.

We can plot the first moment with any conventional plotter, so lets use here `plotly` from `matplotlib`



The first moment here (i.e., the first Kramers—Moyal coefficient) is given solely by the drift term that we have selected `-0.5*x`

And the second moment (i.e., the second Kramers—Moyal coefficient) is a mixture of both the contributions of the diffusive term $b(x)$ and the jump terms ξ and λ.



You have this stored in `moments[2,...]`.

2.3.2 Retrieving the jump-related terms

Naturally one of the most pertinent questions when addressing jump-diffusion processes is the possibility of recovering these same parameters from data. For the given jump-diffusion process we can use the `jump_amplitude` and `jump_rate` functions to non-parametrically estimate the jump amplitude ξ and λ terms.

After having the `moments` in hand, all we need is

```
# first estimate the jump amplitude
xi_est = jd.jump_amplitude(moments = moments)

# and now estimated the jump rate
lamb_est = jd.jump_rate(moments = moments)
```

which resulted in our case in `(xi_est)` ξ `= 2.43` $\pm$ `0.17` and `(lamb_est)` λ `= 1.744 * delta_t` (don't forget to divide `lamb_est` by `delta_t`)!

2.3.3 Other functions and options

Include in this package is also the Milstein scheme of integration, particularly important when the diffusion term has some spacial `x` dependence. `moments` can actually calculate the conditional moments for different lags, using the parameter `lag`.

In `formulae` the set of formulas needed to calculate the second order corrections are given (in `sympy`).

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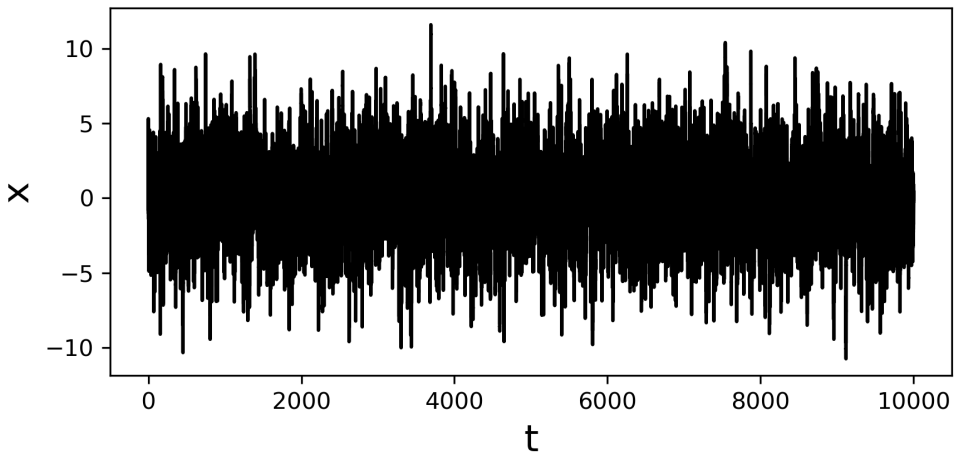

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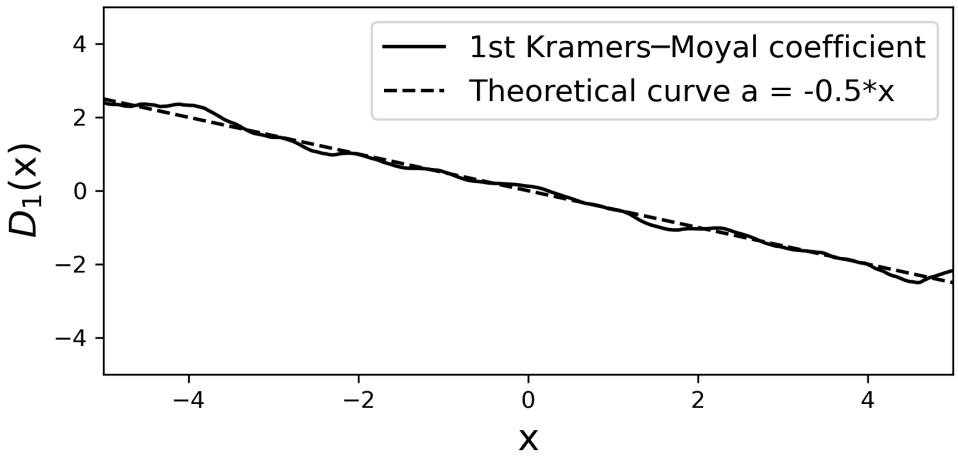

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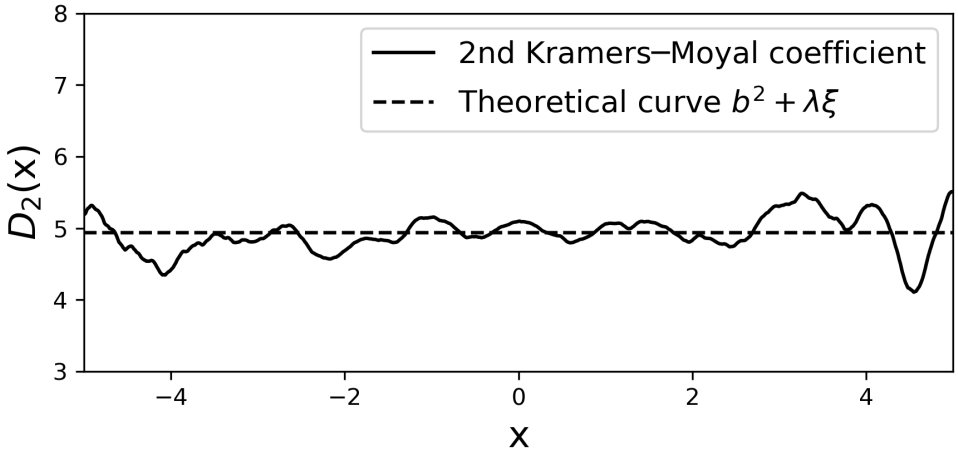

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After having the `moments` in hand, all we need is

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Other functions and options

Include in this package is also the Milstein scheme of integration, particularly important when the diffusion term has some spacial `x` dependence. `moments` can actually calculate the conditional moments for different lags, using the parameter `lag`.

In `formulae` the set of formulas needed to calculate the second order corrections are given (in `sympy`).

3.3 Functions

Documentation for all the functions in `jumpdiff`.

3.3.1 Jump-diffusion timeseries generator

`jumpdiff.jd_process.`**`jd_process`**(*time: float*, *delta_t: float*, *a: callable*, *b: callable*, *xi: float*, *lamb: float*, *init: float = None*, *solver: str = 'Euler'*, *b_prime: callable = None*) → numpy.ndarray

Integrates a jump-diffusion process with drift a(x), diffusion b(x), jump amplitude xi (ξ), and jump rate lamb (λ).

$$\mathrm{d}X(t) = a(x,t)\,\mathrm{d}t + b(x,t)\,\mathrm{d}W(t) + \xi\,\mathrm{d}J(t),$$

with J Poisson with jump rate λ. This integrator has both an Euler–Maruyama and a Milstein method of integration. For Milstein one has to introduce the derivative of the diffusion term `b`, denoted `b_prime`.

Parameters

- **`time`** (*`float > 0`*) – Total integration time. Positive float or int.
- **`delta_t`** (*`float > 0`*) – Time sampling, the smaller the better.
- **`a`** (*`callable`*) – The drift function. Can be a function of a `lambda`. For an Ornstein–Uhlenbeck process with drift `-2x`, a takes the form

  ```
  a = lambda x:  -2x.
  ```

- **`b`** (*`callable`*) – The diffusion function. Can be a function of a `lambda`. For an Ornstein–Uhlenbeck process with diffusion `1`, a takes the form

  ```
  b = lambda x:  1.
  ```

- **`xi`** (*`float > 0`*) – Variance of the jump amplitude, which will be turned into a normal distribution like $\mathcal{N}$`(0,xi)`.
- **`lamb`** (*`float > 0`*) – Jump rate of the Poissonian jumps. This is implemented as the numpy function `np.random.poisson(lam = lamb * delta_t)`.

- **init** (float (defaul None)) – Initial conditions. If None given, generates a random value from a normal distribution ~ $\mathcal{N}$(0, delta_t).
- **solver** (*'Euler' or 'Milstein' (defaul 'Euler')*) – The regular Euler–Maruyama solver 'Euler' is the default, with an order of delta_t. To employ a state-dependent diffusion, i.e., b(x) as a function of x, the Milstein scheme has an order of delta_t. You must introduce as well the derivative of b(x), i.e., b'(x), as the argument b_prime.

Returns **X** – Timeseries of size int(time/delta_t)

Return type np.array

3.3.2 Moments

jumpdiff.moments.**moments**(*timeseries: numpy.ndarray, bw: float = None, bins: numpy.ndarray = None, power: int = 6, lag: list = [1], correction: bool = True, norm: bool = False, kernel: callable = None, tol: float = 1e-10, conv_method: str = 'auto', verbose: bool = False*) → numpy.ndarray

Estimates the moments of the Kramers–Moyal expansion from a timeseries using a Nadaraya–Watson kernel estimator method. These later can be turned into the drift and diffusion coefficients after normalisation.

Parameters

- **timeseries** (*np.ndarray*) – A 1-dimensional timeseries.
- **bw** (*float*) – Desired bandwidth of the kernel. A value of 1 occupies the full space of the bin space. Recommended are values 0.005 < bw < 0.4.
- **bins** (np.ndarray (default None)) – The number of bins for each dimension, defaults to np.array([5000]). This is the underlying space for the Kramers–Moyal conditional moments.
- **power** (int (default 6)) – Upper limit of the the Kramers–Moyal conditional moments to calculate. It will generate all Kramers–Moyal conditional moments up to power.
- **lag** (list (default 1)) – Calculates the Kramers–Moyal conditional moments at each indicated lag, i.e., for timeseries[::lag[]]. Defaults to 1, the shortest timestep in the data.
- **corrections** (bool (default True)) – Implements the second-order corrections of the Kramers–Moyal conditional moments directly
- **norm** (bool (default False)) – Sets the normalisation. False returns the Kramers–Moyal conditional moments, and True returns the Kramers–Moyal coefficients.
- **kernel** (callable (default None)) – Kernel used to convolute with the Kramers–Moyal conditional moments. To select example an Epanechnikov kernel use

  kernel = kernels.epanechnikov

  If None the Epanechnikov kernel will be used.
- **tol** (float (default 1e-10)) – Round to zero absolute values smaller than tol, after convolutions.
- **conv_method** (str (default auto)) – A string indicating which method to use to calculate the convolution. docs.scipy.org/doc/scipy/reference/generated/scipy.signal.convolve.
- **verbose** (bool (default False)) – If True will report on the bandwidth used.

Returns

- **edges** (*np.ndarray*) – The bin edges with shape (D,bins.shape) of the calculated moments.
- **moments** (*np.ndarray*) – The calculated moments from the Kramers–Moyal expansion of the timeseries at each lag. To extract the selected orders of the moments, use `moments[i, :, j]`, with `i` the order according to powers, `j` the lag (if any given).

`jumpdiff.moments.`**`corrections`**(*m: numpy.ndarray*, *power: int*)

The moments function will by default apply the corrections. You can turn the corrections off in that fuction by setting `corrections = False`.

Second-order corrections of the Kramers–Moyal coefficients (conditional moments), given by

$$
\begin{aligned}
F_1 &= M_1, \\
F_2 &= \frac{1}{2}\left(M_2 - M_1^2\right), \\
F_3 &= \frac{1}{6}\left(M_3 - 3M_1M_2 + 3M_1^3\right), \\
F_4 &= \frac{1}{24}\left(M_4 - 4M_1M_3 + 18M_1^2M_2 - 3M_2^2 - 15M_1^4\right), \\
F_5 &= \frac{1}{120}\left(M_5 - 5M_1M_4 + 30M_1^2M_3 - 150M_1^3M_2 + 45M_1M_2^2 - 10M_2M_3 + 105M_1^5\right) \\
F_6 &= \frac{1}{720}\left(M_6 - 6M_1M_5 + 45M_1^2M_4 - 300M_1^3M_3 + 1575M_1^4M_2 - 675M_1^2M_2^2\right. \\
&\qquad \left. +180M_1M_2M_3 + 45M_2^3 - 15M_2M_4 - 10M_3^2 - 945M_1^6\right),
\end{aligned}
$$

with the prefactor the normalisation, i.e., the normalised results are the Kramers–Moyal coefficients. If `norm` is False, this results in the Kramers–Moyal conditional moments.

Parameters

- **`(moments)`** (*m*) – The calculated conditional moments from the Kramers–Moyal expansion of the at each lag. To extract the selected orders of the moments use `moments[i,:, j]`, with `i` the order according to powers, `j` the lag.
- **`power`** (*int*) – Upper limit of the Kramers–Moyal conditional moments to calculate. It will generate all Kramers–Moyal conditional moments up to power.

Returns **F** – The corrections of the calculated Kramers–Moyal conditional moments from the Kramers–Moyal expansion of the timeseries at each lag. To extract the selected orders of the moments, use `F[i, :, j]`, with `i` the order according to powers, `j` the lag (if any introduced).

Return type np.ndarray

3.3.3 Parameters

`jumpdiff.parameters.`**`jump_amplitude`**(*moments: numpy.ndarray*, *tol: float = 1e-10*, *full: bool = False*, *verbose: bool = False*) → numpy.ndarray

Retrieves the jump amplitude xi (ξ) via

$$\lambda(x,t) = \frac{M_4(x,t)}{3\sigma_\xi^4}.$$

Take notice that the different normalisation of the `moments` leads to a different results.

Parameters

- **`moments`** (*np.ndarray*) – Moments extracted with the function `moments`. Needs moments up to order `6`.

- **tol** (float (defaul `1e-10`)) – Toleration for the division of the moments.
- **full** (bool (defaul `False`)) – If `True` returns also the (biased) weighed standard deviation of the averaging process.
- **verbose** (bool (defaul `True`)) – Prints the result.

Returns **xi_est** – Estimator of the jump amplitude xi (ξ).

Return type np.ndarray

References

Anvari, M., Tabar, M. R. R., Peinke, J., Lehnertz, K., 'Disentangling the stochastic behavior of complex time series.' Scientific Reports, 6, 35435, 2016. doi: 10.1038/srep35435.

Lehnertz, K., Zabawa, L., and Tabar, M. R. R., 'Characterizing abrupt transitions in stochastic dynamics.' New Journal of Physics, 20(11):113043, 2018. doi: 10.1088/1367-2630/aaf0d7.

jumpdiff.parameters.**jump_rate**(*moments: numpy.ndarray*, *xi_est: numpy.ndarray = None*, *tol: float = 1e-10*, *full: bool = False*, *verbose: bool = False*) → numpy.ndarray

Retrieves the jump rate lamb (λ) via

$$\sigma_\xi^2 = \frac{M_6(x,t)}{5M_4(x,t)}.$$

Take notice that the different normalisation of the `moments` leads to a different results.

Parameters

- **moments** (*np.ndarray*) – moments extracted with the function 'moments'. Needs moments of order 6.
- **tol** (float (defaul `1e-10`)) – Toleration for the division of the moments.
- **full** (bool (defaul `False`)) – If `True` returns also the (biased) weighed standard deviation of the averaging process.
- **verbose** (bool (defaul `True`)) – Prints the result.

Returns **xi_est** – Estimator on the jump rate lamb (λ)

Return type np.ndarray

References

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3.3.4 Q-ratio

jumpdiff.q_ratio.**q_ratio**(*lag: numpy.ndarray*, *timeseries: numpy.ndarray*, *loc: int = None*, *correction: bool = False*) → numpy.ndarray

q_ratio method to distinguish pure diffusion from jump-diffusion timeseries, Given by the relation of the 4th

and 6th Kramers–Moyal coefficient with increasing lag

$$Q(x,\tau) = \frac{D_6(x,\tau)}{5D_4(x,\tau)} = \begin{cases} b(x)^2\tau, & \text{diffusive} \\ \sigma_\xi^2(x), & \text{jumpy} \end{cases}$$

Parameters

- **lag** (*np.ndarray of ints*) – An array with the time-lag to extract the Kramers–Moyal coefficient for different lags.
- **timeseries** (*np.ndarray*) – A 1-dimensional timeseries.
- **loc** (float (defaul `None`)) – Use a particular point in space to calculate the ratio. If `None` given, the maximum of the probability density function is taken.
- **corrections** (bool (defaul `False`)) – Select whether to use corrective terms.

Returns

- **lag** (*np.ndarray of ints*) – Same as input, but only lag > 0 and as ints.
- **ratio** (*np.ndarray of len(lag)*) – Ratio of the sixth-order over forth-order Kramers–Moyal coefficient.

References

Anvari, M., Tabar, M. R. R., Peinke, J., Lehnertz, K., 'Disentangling the stochastic behavior of complex time series.' Scientific Reports, 6, 35435, 2016. doi: 10.1038/srep35435.

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3.3.5 Formulae

`jumpdiff.formulae.`**`m_formula`**(*power*, *tau=True*)

Generate the formula for the conditional moments with second-order corrections based on the relation with the ordinary Bell polynomials

$$M_n(x',\tau) \sim (n!)\tau D_n(x') + \frac{(n!)\tau^2}{2}\sum_{m=1}^{n-1} D_m(x')D_{n-m}(x')$$

Parameters **power** (*int*) – Desired order of the formula.

Returns **term** – Expression up to given `power`.

Return type sympy.symbols

`jumpdiff.formulae.`**`f_formula`**(*power*)

Generate the formula for the conditional moments with second-order corrections based on the relation with the ordinary Bell polynomials

$$D_n(x) = \frac{1}{\tau(n!)}\Big[\hat{B}_{n,1}\left(M_1(x,\tau), M_2(x,\tau), \ldots, M_n(x,\tau)\right) - \frac{\tau}{2}\hat{B}_{n,2}\left(M_1(x,\tau), M_2(x,\tau), \ldots, M_{n-1}(x,\tau)\right)\Big].$$

Parameters **power** (*int*) – Desired order of the formula.

Returns **term** – Expression up to given `power`.

Return type sympy.symbols

jumpdiff.formulae.**f_formula_solver**(*power*)
Generate the reciprocal relation of the moments to the Kramers–Moyal coefficients by sequential iteration.

$$D_n(x) = \frac{1}{\tau(n!)}\left[\hat{B}_{n,1}\left(M_1(x,\tau), M_2(x,\tau), \ldots, M_n(x,\tau)\right) - \frac{\tau}{2}\hat{B}_{n,2}\left(M_1(x,\tau), M_2(x,\tau), \ldots, M_{n-1}(x,\tau)\right)\right].$$

Parameters **power** (*int*) – Desired order of the formula.

Returns **term** – Expression up to given power.

Return type sympy.symbols

3.3.6 Helping functions

Kernels function

jumpdiff.kernels.**kernel**(*kernel_func*)
Transforms a kernel function into a scaled kernel function (for a certain bandwidth bw).

Currently implemented kernels are: Epanechnikov, Gaussian, Uniform, Triangular, Quartic.

For a good overview of various kernels see https://en.wikipedia.org/wiki/Kernel_(statistics)

jumpdiff.kernels.**volume_unit_ball**(*dims: int*) → float
Returns the volume of a unit ball in dimensions dims.

jumpdiff.kernels.**epanechnikov**(*x: numpy.ndarray*, *dims: int*) → numpy.ndarray
The Epanechnikov kernel in dimensions dims.

jumpdiff.kernels.**gaussian**(*x: numpy.ndarray*, *dims: int*) → numpy.ndarray
Gaussian kernel in dimensions dims.

jumpdiff.kernels.**uniform**(*x: numpy.ndarray*, *dims: int*) → numpy.ndarray
Uniform, or rectangular kernel in dimensions dims

jumpdiff.kernels.**triagular**(*x: numpy.ndarray*, *dims: int*) → numpy.ndarray
Triagular kernel in dimensions dims

jumpdiff.kernels.**quartic**(*x: numpy.ndarray*, *dims: int*) → numpy.ndarray
Quartic, or biweight kernel in dimensions dims

3.4 License

3.5 Contact

If you need help with something, find a bug, issue, or typo on the repository or in the code, you can contact me here: leonardo.rydin@gmail.com or open an issue on the GitHub repository.

CHAPTER 4

Literature

[1] Tabar, M. R. R. *Analysis and Data-Based Reconstruction of Complex Nonlinear Dynamical Systems.* Springer, International Publishing (2019), Chapter Stochastic Processes with Jumps and Non-vanishing Higher-Order Kramers—Moyal Coefficients*.

[2] Friedrich, R., Peinke, J., Sahimi, M., Tabar, M. R. R. *Approaching complexity by stochastic methods: From biological systems to turbulence,* Physics Reports 506, 87–162 (2011).

[3] Rydin Gorjão, L., Meirinhos, F. *kramersmoyal: Kramers–Moyal coefficients for stochastic processes.* Journal of Open Source Software, 4(44) (2019).

An extensive review on the subject can be found here.

CHAPTER 5

Funding

Helmholtz Association Initiative *Energy System 2050 - A Contribution of the Research Field Energy* and the grant No. VH-NG-1025 and *STORM - Stochastics for Time-Space Risk Models* project of the Research Council of Norway (RCN) No. 274410.

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